

RATIONAL PARAMETRIZATION OF LEMNISCATES

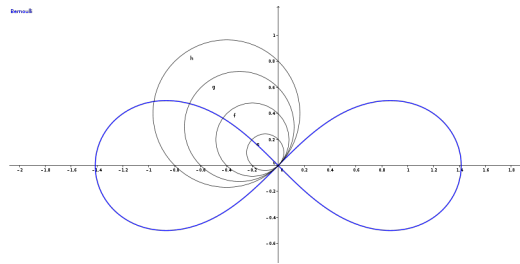
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1 Lemniscate of Bernoulli

The implicit equation for the Lemniscate of Bernoulli is given by:

$$(x^2 + y^2)^2 - 2(x^2 - y^2) = 0 \quad (1)$$



Parametrize with a substitution using a pencil of circles through (0,0):

$$x^2 + y^2 = t(x - y) \quad (2)$$

Substituting 2 into 1 and expanding the difference of two squares:

$$t^2(x - y)^2 - 2(x - y)(x + y) = 0$$

$$(x - y)(t^2(x - y) - 2(x + y)) = 0$$

Solving for the second term as it contains t :

$$t^2x - 2x - t^2y - 2y = x(t^2 - 2) - y(t^2 + 2) = 0$$

$$y = x \frac{t^2 - 2}{t^2 + 2}$$

Substitute $y = x \frac{t^2 - 2}{t^2 + 2}$ in 2:

$$x^2 + x^2 \frac{(t^2 - 2)^2}{(t^2 + 2)^2} - xt + x \frac{t(t^2 - 2)}{t^2 + 2} = 0$$

$$x \left(x + x \frac{(t^2 - 2)^2}{(t^2 + 2)^2} - t + \frac{t(t^2 - 2)}{t^2 + 2} \right) = 0$$

Which has solutions $x = 0$ and x as a function of t :

$$x \left(1 + \frac{(t^2 - 2)^2}{(t^2 + 2)^2} \right) = t - \frac{t(t^2 - 2)}{t^2 + 2}$$

$$x \left(\frac{(t^2 + 2)^2 + (t^2 - 2)^2}{(t^2 + 2)^2} \right) = \frac{t(t^2 + 2) - t(t^2 - 2)}{t^2 + 2}$$

$$x = \frac{(t^2 + 2)^2}{t^4 + 4t^2 + 4 + t^4 - 4t^2 + 4} \bullet \frac{t^3 + 2t - t^3 + 2t}{t^2 + 2}$$

$$x = \frac{4t(t^2 + 2)}{2t^4 + 8}$$

$$y = \frac{2t(t^2 + 4)}{t^4 + 4} \bullet \frac{t^2 - 2}{t^2 + 2}$$

A rational parametrization of the Lemniscate of Bernoulli is:

$$x(t) = \frac{2t(t^2 + 2)}{t^4 + 4}$$

$$y(t) = \frac{2t(t^2 - 2)}{t^4 + 4}$$

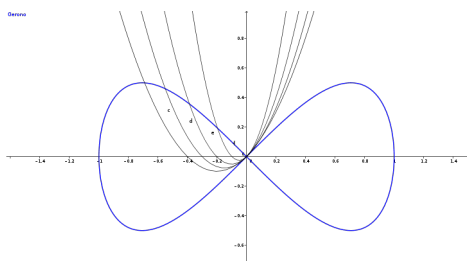
2 Lemniscate of Geron

The lemniscate of Geron is given by the implicit equation:

$$x^4 - a^2(x^2 - y^2) = 0 \tag{3}$$

Parametrize with a substitution using a pencil of parabolas through (0,0):

$$x^2 = t(x - y) \tag{4}$$



Substituting 4 into 3 and expanding the difference of two squares:

$$t^2(x-y)^2 - a^2(x-y)(x+y) = 0$$

$$(x-y)(t^2(x-y) - a^2(x+y)) = 0$$

Solving for the second term as it contains t :

$$x(t^2 - a^2) - y(t^2 + a^2) = 0$$

So we can substitute $y = x \frac{t^2 - a^2}{t^2 + a^2}$ in 4:

$$x^2 - t \left(x - x \frac{t^2 - a^2}{t^2 + a^2} \right) = 0$$

$$x^2 - x \left(\frac{t(t^2 + a^2) - t(t^2 - a^2)}{t^2 + a^2} \right) = 0$$

$$x \left(x - \frac{2a^2t}{t^2 + a^2} \right) = 0$$

Which has solutions $x = 0$ and $x = \frac{2a^2t}{t^2 + a^2}$. Then:

$$y = \frac{2a^2t}{t^2 + a^2} \cdot \frac{t^2 - a^2}{t^2 + a^2}$$

A rational parametrization for the Lemniscate of Gerono is:

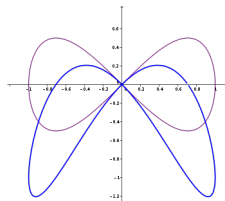
$$x(t) = \frac{2a^2t}{t^2 + a^2}$$

$$y(t) = \frac{2a^2t(t^2 - a^2)}{(t^2 + a^2)^2}$$

3 Cramer's Besace

The Besace is a lemniscate with coefficient a in a position that makes the parametrization method less obvious.

$$x^4 - x^2 + (y^2 + ax^2)^2 = 0 \tag{5}$$



Note that when $a = 0$, we have the lemniscate of Geronno of 3 with $a = 1$. However, using the same initial substitution doesn't work well and another approach is required.

Rearrange 4 to collect terms of the same order:

$$f(x, y) = (a^2 + 1)x^4 + 2ax^2y + (y^2 - x^2)$$

$$f = f_d + f_{d-1} + f_{d-2}$$

Parametrization is a two step process, first in terms of τ and then the more specific τ_0 in terms of a polynomial in t such that the polynomial ensures a discriminant becomes a perfect square.

First, define $y = \tau x$ and solve

$$F(1, \tau) = x^2 f_4(1, \tau) + x f_3(1, \tau) + f_2(1, \tau)$$

$$F(1, \tau) = (a^2 + 1)x^2 + 2a\tau x + \tau^2 - 1$$

$$D(\tau) = (2a\tau)^2 - 4(a^2 + 1)^2(\tau^2 - 1)$$

$$D(\tau) = 4(a^2 + 1 - \tau^2)$$

$D(\tau) = P^2 \cdot Q(\tau)$ where $Q(\tau) = \alpha + \beta\tau + \gamma\tau^2$, $\alpha = (a^2 + 1)$, $\beta = 0$, $\gamma = -1$ allows the rationalizer to be set with parameter t :

$$\tau_0 = \frac{4(a^2 + 1)t}{t^2 + 4(a^2 + 1)}$$

$D(\tau_0) = P^2 \cdot Q(\tau_0) = P^2 \cdot R^2(\tau_0)$, then

$$x = \frac{\frac{-2a \cdot 4(a^2 + 1)t}{t^2 + 4(a^2 + 1)} + P \cdot R(\tau_0)}{2(a^2 + 1)}$$

$$x = \frac{\frac{-2a \cdot 4(a^2 + 1)t}{t^2 + 4(a^2 + 1)} + 2 \frac{\sqrt{(a^2 + 1)(t^2 - 4(a^2 + 1))}}{t^2 + 4(a^2 + 1)}}{2(a^2 + 1)}$$

A parametrization for the Besace is

$$x(t) = \frac{\frac{1}{\sqrt{(a^2 + 1)}}t^2 - 4at - 4\sqrt{a^2 + 1}}{t^2 + 4(a^2 + 1)}$$

$$y(t) = \frac{4(a^2 + 1)t}{t^2 + 4(a^2 + 1)} \bullet \frac{\frac{1}{\sqrt{(a^2 + 1)}}t^2 - 4at - 4\sqrt{a^2 + 1}}{t^2 + 4(a^2 + 1)}$$